LOWER BOUNDS FOR GENERALIZED UPCROSSINGS OF ERGODIC AVERAGES

S. E. FERRANDO, P. J. CATUOGNO, AND A. L. GONZALEZ

Abstract. New lower bound inequalities are obtained for generalized upcrossings of ergodic averages. Results and techniques are presented in such a way that the duality with Bishop's results on upper bounds is emphasized. Finally, the significance of generalized upcrossings as a mean to count spatial oscillations is clarified.

1. Introduction

Upcrossing inequalities (u.i.) are a basic phenomena for ergodic averages, in particular, they imply the ergodic theorem. Bishop established upper bounds for (generalized) upcrossings in [2] and [3] by using two different techniques and in a more general setting than the one of ergodic averages. Our results complement those of Bishop by proving lower bounds for generalized upcrossings in the setting of measure preserving transformations and Cesaro averages. The main motivation to study lower bounds is that they give information on the number of spatial oscillations for the ergodic averages. Lower bounds, in the form of reverse inequalities, have been studied in [5] but with a different perspective.

We now describe a result in our paper which establishes a strikingly thight inequality, detailed definitions are presented elsewhere. Let $w_{\eta,\alpha,n}(x)$ denote the number of generalized upcrossings up to time n with respect to function f and transformation τ. Set $w_{\eta,\alpha}(x) = \sup_n w_{\eta,\alpha,n}(x)$, a constructive result of Bishop implies (using classical arguments) the following result:

$$\int \eta \; w_{\eta,\alpha}(x) \leq \int (f-\alpha)_+ .$$

Under appropriate conditions, our Theorem 2 shows,

$$\int (f-\alpha-\eta)_+ \leq \int \eta \; w_{\eta,\alpha}(x).$$

We remark on the fact that our result, in contrast to Bishop's, requires $w_{\eta,\alpha}(x)$ and it is not possible to obtain a similar result using the finite time quantity $w_{\eta,\alpha,n}(x)$.

The paper is organized as follows, in Section 2 we introduce the main definitions and proceed to prove the basic counting inequalities. We present new results on lower bounds along with Bishop's results on upper bounds (as presented in [3]), this can be done with few extra efforts and in this way we emphasize the relationship and novelty of our arguments in contrast to the ones of Bishop. Section 3 introduces the concepts and intermediate results needed to integrate the pointwise

Date: June 8, 2002.

1991 *Mathematics Subject Classification.* Primary:28D99 ; secondary:47A35.

Key words and phrases. Cesaro averages. Ergodic theorem. Upcrossing inequalities. Pointwise a.e. convergence.

Research partially supported by NSERC.

inequalities from Section 2; our main result, Theorem 2, is then proved. Section 4 clarifies and draws connections between generalized upcrossings and other measures of spatial oscillations. Furthermore, Proposition 2 gives information on the pointwise asymptotics of generalized upcrossings. For completeness, the brief Section 5 states the dual results for downcrossings. Finally, Appendix A states, for the reader's convenience, a known result needed in the main text.

2. Pointwise Inequalities for Generalized Upcrossings

We adopt the convention that pointwise inequalities, not containing explicit quantifiers refering to the point x, are valid for all values of x where the quantities involved are defined. Given our settings, this will imply almost everywhere (a.e.) on the measure space.

Definition 1. Given an integer $n \geq 0$, the sequence $P = (s_1, t_1, \ldots, s_m, t_m)$ is called *n-admissible* if $-1 \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_m < t_m \leq n$. We let $|P| = m$ denote the *size* of P. The finite set of all n-admissible sequences is denoted by $\mathcal{P}^n$. We allow the void sequence $P = \emptyset$ and define $|P| = 0$ in this case.

Definition 2. Let $a_{-1}, a_0, \ldots, a_n$ and $b_{-1}, b_0, \ldots, b_n$ be given real numbers. A sequence $P = (u_1, v_1, \ldots, u_N, v_N)$ is called an *n-crossing sequence* if it is an n-admissible sequence which satisfies

$$(1)\qquad \begin{aligned} a_{u_i} &\leq b_{v_i},\ i = 1, \ldots, N, \\ a_{u_{i+1}} &\leq b_{v_i},\ i = 1, \ldots, N-1. \end{aligned}$$

Thus a crossing sequence is a special kind of admissible sequence. The finite set of all n-crossing sequences is denoted by $\mathcal{P}_0^n$.

For a nonvoid admissible sequence $P = (s_1, t_1, \ldots, s_m, t_m)$, we define:

$$(2)\qquad S(P) = \sum_{i=1}^{|P|} (b_{t_i} - a_{s_i}).$$

If P is void we define $S(P) = 0$. Let $\mathcal{P}_1^n$ be the set of n-admissible sequences P_1 with $S(P_1)$ maximal in $\mathcal{P}^n$, i.e. the maximum is taken over $\mathcal{P}^n$. Let $\mathcal{P}_2^n$ be the set of sequences P_2 in $\mathcal{P}_1^n$ with $|P_2|$ maximal in $\mathcal{P}_1^n$.

The next lemma is essentially contained in Lemma 6, [3], pp. 234-235, we have added material needed to prove lower bounds and we re-wrote the lemma to serve our needs.

Lemma 1. *The following two statements hold for any $n \geq 0$:*
i) $\mathcal{P}_1^n \subseteq \mathcal{P}_0^n$.
ii) If $P_2 \in \mathcal{P}_2^n$, then:

$$|P_2| \geq |P| \text{ for all } P \in \mathcal{P}_0^n.$$

Proof. i) Let $P_1 = (s_1, t_1, \ldots, s_m, t_m)$ belong to $\mathcal{P}_1^n$. If $P_1 \notin \mathcal{P}_0^n$, there would exist i, $1 \leq i \leq m$ such that $a_{s_i} > b_{t_i}$ or $1 \leq i \leq m-1$ such that $a_{s_{i+1}} > b_{t_i}$, then deleting (s_i, t_i) or (t_i, s_{i+1}) from P_1 we obtain an n-admissible sequence Q with $S(Q) > S(P_1)$, which contradicts its maximality. Thus $\mathcal{P}_1^n \subseteq \mathcal{P}_0^n$.

ii) Let $P_2 = (s_1, t_1, \ldots, s_m, t_m)$ belong to $\mathcal{P}_2^n$ and $P = (u_1, v_1, \ldots, u_N, v_N) \in \mathcal{P}_0^n$. It is not possible that there exist $i \in \{1, \cdots, N-1\}$ and $j \in \{1, \cdots, m\}$ such that

$$(3)\qquad s_j < v_i \leq u_{i+1} < t_j,$$

otherwise, if (3) holds, $P' = (s_1, t_1, \dots, s_j, v_i, u_{i+1}, t_j \dots s_m, t_m)$ would be n-admissible with $|P'| = |P_2| + 1$ and

$$\begin{aligned} S(P') &= \sum_{k=1}^{j-1} \left(b_{t_j} - a_{s_j}\right) + b_{v_i} - a_{s_j} + b_{t_j} - a_{u_{i+1}} + \sum_{k=j+1}^{m} \left(b_{t_j} - a_{s_j}\right) = \\ &= S(P) + b_{v_i} - a_{u_{i+1}} \geq S(P), \end{aligned}$$

which contradicts the fact that $P_2 \in \mathcal{P}_2^n$.
For convenience set $t_0 = -1$ and $s_{m+1} = n$, with a similar argument, we see that it is neither possible that, for any $i = 1, \cdots, N$ and any $j = 0, \cdots m$,

$$t_j \leq u_i < v_i \leq s_{j+1}. \tag{4}$$

Let us note that $u_i < t_m$ for all $i = 1, \cdots, N$. Thus for each $i = 1, \cdots, N$ there exist $j = j(i) = 1, \cdots, m$ such that,

$$u_i \in [t_{j-1}, t_j).$$

Now if $i_1 < i_2$, we have : $t_{j(i_1)-1} \leq u_{i_1}$ and having in mind that (3) and (4) do not hold

$$s_{j(i_1)} < v_{i_1}, \text{ and then } \quad t_{j(i_1)} \leq u_{i_2}.$$

This shows that the map $i \to j(i)$ is injective, thus $N \leq m$. □

We will especialize the general setting just introduced to the following situation: for a given function (real valued and measurable) $f(x)$, τ a measurable point transformation on a measure space $(X, \mathcal{F}, \mu)$ we let $f_j(x) = T^j f(x) = f(\tau^j x)$ (so $f_0 = f$). Define also $\mathcal{I} = \{A \in \mathcal{F} : \tau^{-1}(A) = A\}$, $A \in \mathcal{I}$ is called an invariant subset. We use the notation $A_t f(x) = 1/(t+1) \sum_{j=0}^{t} f(\tau^j x)$ and set $A_{-1} f(x) = 0$ for all x. For real numbers α, η ($\eta > 0$) and given x we especialize the fixed finite sequences $\{a_i\}$ and $\{b_i\}$ in Definition 2 as follows,

$$b_t = b_t(x) = b_{t,\eta,\alpha}(x) = \sum_{j=0}^{t} \left(f_j(x) - \alpha - \eta\right)$$

and

$$a_s = a_s(x) = a_{s,\alpha}(x) = \sum_{j=0}^{s} \left(f_j(x) - \alpha\right).$$

Of course, in the expressions above, a sum over a void set has the value 0. In a natural way we augment the notations introduced earlier by making explicit reference to the point x and possibly other parameters. For example, if $P = (s_1, t_1, \dots, s_m, t_m) \in \mathcal{P}^n$ we also define $S(P)(x) = \sum_{j=1}^{m} \left(b_{t_j}(x) - a_{s_j}(x)\right)$, then $\mathcal{P}_1^n(x, \eta, \alpha)$ (more compactly, $\mathcal{P}_1^n(x)$) are the elements $P \in \mathcal{P}^n$ such that $S(P)(x)$ is maximal. In particular, $\mathcal{P}_0^n$ especializes to $\mathcal{P}_0^n(x, \eta, \alpha)$, an element P of this set will be called an *n-generalized upcrossing sequence at* x. The reasons for the switch from the use of *crossing* to *generalized upcrossing* is given by Proposition 1 (see also remarks above Corollary 1) and our use of crossings in Section 5. We will freely hide some of the parameters (mainly α and η) whenever possible. The following notation will be used

$$\lambda_{\eta,\alpha,n}(x) = \max_{P \in \mathcal{P}^n} S(P)(x) = S(P_1)(x), \tag{5}$$

where P_1 is any element in $\mathcal{P}_1^n(x)$.
For the rest of the paper, except Section 5, the above conventions and assumptions will be used freely without necessarily making them explicit.

Definition 3. For given integer $n \geq 0$, define the (maximum) *number of n-generalized upcrossings* at x by

$$w_{\eta,\alpha,n}(x) = \max\{|P| : \text{where } P \text{ is an n-generalized upcrossing sequence at } x\}.$$

Also, define the number of generalized upcrossings at x by $w_{\eta,\alpha}(x) = \lim_{n\to\infty} w_{\eta,\alpha,n}(x)$, without further restrictions nothing forbids $w_{\eta,\alpha}(x)$ being infinity at this moment. Our use of the word *generalized* is in order to distinguish $w_{\eta,\alpha,n}(x)$ from the usual (geometric) upcrossings from Definition 4.

Lemma 1 implies the following corollary.

Corollary 1.

$$|P_2| = w_{\eta,\alpha,n}(x) \text{ and } \lambda_{\eta,\alpha,n}(x) = S(P_2)(x), \tag{6}$$

where P_2 is any sequence in $\mathcal{P}_2^n(x)$.

The following lemma is key to obtain Theorem 2 on lower bounds.

Lemma 2. *For all $n \geq 1$ the following holds:*

$$\lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) \leq \eta\ w_{\eta,\alpha,n-1}(\tau x) - (f(x) - \alpha - \eta)_+. \tag{7}$$

Proof. Consider $P' = (s'_1, t'_1, \ldots, s'_{m'}, t'_{m'}) \in \mathcal{P}^{n-1}$ to be nonempty and let $s_i = s'_i + 1$, $t_i = t'_i + 1$ for $i = 1, \ldots, m'$, this defines $P = (s_1, t_1, \ldots, s_{m'}, t_{m'}) \in \mathcal{P}^n$ with $s_1 \geq 0$. Then

$$S(P')(\tau x) - \eta\ m' = S(P)(x) \leq \lambda_{\eta,\alpha,n}(x) - (f(x) - \alpha - \eta)_+, \tag{8}$$

the equality in (8) can be checked directly, to argue for the inequality in (8) notice that if $(f(x) - \alpha - \eta) \leq 0$ the inequality holds. On the other hand, if $(f(x) - \alpha - \eta) \geq 0$, define $Q = (s_0 = -1, t_0 = 0, s_1, t_1, \ldots, s_{m'}, t_{m'}) \in \mathcal{P}^n$ and notice that $S(P)(x) = S(Q)(x) - (f(x) - \alpha - \eta) \leq \lambda_{\eta,\alpha,n}(x) - (f(x) - \alpha - \eta)_+$. If $P' = \emptyset$ equation (8) also holds due to $\lambda_{\eta,\alpha,n}(x) \geq (f(x) - \alpha - \eta)_+$. Therefore, given that P' is an arbitrary element of $\mathcal{P}^{n-1}$, evaluating (8) at $P' \in \mathcal{P}_2^{n-1}(\tau x)$ and using (6) we obtain

$$\lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) \leq \eta\ w_{\eta,\alpha,n-1}(\tau x) - (f(x) - \alpha - \eta)_+, \tag{9}$$

which is valid for any $n \geq 1$. □

The next result complements the previous lemma, it can be found, in a more general setting, in [3].

Lemma 3.

$$\eta\ w_{\eta,\alpha,n}(x) \leq \lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) + (f(x) - \alpha)_+. \tag{10}$$

Proof. Consider $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}^n$. Now define $P' = (s'_1, t'_1, \ldots, s'_{m'}, t'_{m'}) \in \mathcal{P}^{n-1}$ as follows: if $t_1 > 0$ and $m \geq 1$ let $t'_i = t_i - 1,\ s'_i = s_i - 1$ for $i = 1, \ldots, m$ and $m' = m$ with the understanding that $s'_1 = -1$ if $s_1 = -1$. If $t_1 = 0$ and $m \geq 2$ take $m' = m - 1$ and $t'_i = t_{i+1} - 1,\ s'_i = s_{i+1} - 1$ for $i = 1, \ldots, m'$. In case $t_1 = 0$ and $m = 1$ take $P' = \emptyset$. We then obtain,

$$S(P)(x) \leq S(P')(\tau x) + (f(x) - \alpha)_+ - \eta\, m \leq \lambda_{\eta,\alpha,n-1}(\tau x) + (f(x) - \alpha)_+ - \eta\, m. \tag{11}$$

Then use Corollary 1 to complete the proof. □

Let $\chi_A(x)$ denote the characteristic function of a set A.

Lemma 4. *For all $n \geq 1$ the following holds:*

(12)
$$w_{\eta,\alpha,n}(x) \leq w_{\eta(1+1/n),\alpha,n-1}(\tau x) + \chi_{\{f(x)-\alpha\geq\eta(1-1/n)\}}(x)\ \chi_{\{f(\tau x)-\alpha\leq\eta(1+1/n)\}}(x).$$

Proof. We start with the following *observation*: given nonnegative integers u, v, if $b_{v,\eta}(x) \geq a_u(x)$ it follows that $b_{v-1,\eta(1+1/n)}(\tau x) \geq a_{u-1}(\tau x)$. Next consider $P(x) = \{-1 \leq s_1(x) < t_1(x) \leq s_2(x) < \ldots < t_m(x) \leq n\} \in \mathcal{P}_0^n(x,\eta)$ with $m = w_{\eta,\alpha,n}(x)$ (we will supress the x in $t_i(x)$ and $s_i(x)$ when convenient). The above observation implies,

$$w_{\eta,\alpha,n}(x) \leq w_{\eta(1+1/n),\alpha,n-1}(\tau x) + \chi_{\{s_1(x)=-1\}}(x). \tag{13}$$

Therefore, it is enough to consider $s_1(x) = -1$ for the rest of the proof. To simplify the notation let $A = \{f(x) - \alpha \geq \eta(1-1/n)\}$ and $B = \{f(\tau x) - \alpha < \eta(1+1/n)\}$. Consider the following cases: Case I) when $\chi_B(x) = 0$. Then, we should have $s_2(x) \geq 2$, otherwise, we obtain a contradiction with the upcrossing condition $-\sum_{j=0}^{s_2}(f(\tau^j x) - \alpha) + \sum_{j=0}^{t_1}(f(\tau^j x) - \alpha - \eta) \geq 0$. Then define $P' = \{-1 \leq s'_1 < t'_1 < s'_2 < \ldots < t'_m \leq n-1\}$ by $t'_i = t_i - 1$, $s'_i = s_i - 1$ for $i \geq 2$, $s'_1 = -1$ and $t'_1 = t_1 - 1$ if $\sum_{j=0}^{t_1-1}(f(\tau^{j+1}x) - \alpha - \eta(1+1/n)) \geq 0$ otherwise let $t'_1 = 0$. We claim that

$$P' \in \mathcal{P}_0^{n-1}(\tau x, \eta(1+1/n)), \tag{14}$$

and hence in the present case I), (14) will prove (12). Given the *observation* mentioned above and the fact that $P \in \mathcal{P}_0^n(x,\eta)$, it follows that in order to check (14) we only need to see if $b_{t'_1,\eta(1+1/n)}(\tau x) \geq 0$. But this holds from our choice of t'_1. Consider now Case II), namely $\chi_B(x) = 1$, it follows from (13) that it is enough to consider $\chi_A(x) = 0$. We remark that under this condition we have $t_1(x) \geq 1$, hence we can define $P' = \{-1 \leq s'_1 < t'_1 < s'_2 < \ldots < t'_m \leq n-1\}$ by $t'_i = t_i - 1$, $s'_i = s_i - 1$ for $i \geq 2$, $s'_1 = -1$ and $t'_1 = t_1 - 1$. We claim that $P' \in \mathcal{P}_0^{n-1}(\tau x, \eta(1+1/n))$, and hence the proof of (12) will be complete. From the observation at the beginning of the proof and the fact that $s_1 = -1$ we only need to check if $b_{t'_1,\eta(1+1/n)}(\tau x) \geq 0$. But, due to $P \in \mathcal{P}_0^n(x,\eta)$ we have the following

$$0 \leq b_{t_1,\eta}(x) = b_{t'_1,\eta}(\tau x) + f(x) - \alpha - \eta = b_{t'_1,\eta(1+1/n)}(\tau x) + f(x) - \alpha - \eta(1+1/n), \tag{15}$$

hence $b_{t'_1,\eta(1+1/n)}(\tau x) \geq 0$ follows from $\chi_A(x) = 0$. □

The following lemma complements Lemma 4.

Lemma 5. *Fix α and η be real numbers ($\eta > 0$), then for all $\eta' \leq \eta/2$ and $n \geq 1$:*

$$w_{\eta',\alpha,n}(x) \geq w_{\eta,\alpha,n-1}(\tau x)\ + \chi_{\{f(\tau x)-\alpha\leq-\eta'\}}(x)\ \chi_{\{f(x)-\alpha\geq\eta'\}}(x). \tag{16}$$

Proof. We start with the following *observation*: given $-1 \leq u_1 < v \leq u_2$. If $b_{v,\eta}(\tau x) \geq a_{u_1}(\tau x)$ and $b_{v,\eta}(\tau x) \geq a_{u_2}(\tau x)$, using $\eta' \leq \eta/2$, it follows that $b_{v+1,\eta'}(x) \geq a_{u_1+1}(x)$ and $b_{v+1,\eta'}(x) \geq a_{u_2+1}(x)$.

We may assume $w_{\eta,\alpha,n-1}(\tau x) \geq 1$ all along the proof, then, from Corollary 1 there exists $P \in \mathcal{P}_2^{n-1}(\tau x,\eta)$, $P = \{-1 \leq s_1 < t_1 \leq \ldots < t_m \leq n-1\}$ and $m = |P| = w_{\eta,\alpha,n-1}(\tau x)$. Also from Lemma 1 part i), $P \in \mathcal{P}_0^{n-1}(\tau x,\eta)$. Define $P' = \{-1 < s'_1 < t'_1 \leq \ldots < t'_m \leq n\}$ by $s'_i = s_i + 1, t'_i = t_i + 1$, $i = 1, \ldots, m$. The above *observation* implies $P' \in \mathcal{P}_0^n(x,\eta')$, then, due to $|P'| = m = w_{\eta,\alpha,n-1}(\tau x)$, it follows that $w_{\eta',\alpha,n}(x) \geq w_{\eta,\alpha,n-1}(\tau x)$. Therefore, to establish (16) we may assume

x satisfies $(f(\tau x) - \alpha) \leq -\eta'$ and $(f(x) - \alpha) \geq \eta'$ for the rest of the proof. The inequality $(f(\tau x) - \alpha) \leq -\eta'$ implies:

$$s_1 \geq 0 \text{ and } -(\sum_{j=0}^{s_1}(f(\tau^{j+1}x) - \alpha)) \geq -(f(\tau x) - \alpha) \geq \eta'. \tag{17}$$

To prove the statements in (17), notice that $s_1 = -1$ and $f(\tau x) - \alpha - \eta \leq -\eta' - \eta < 0$ contradict the fact that $P \in \mathcal{P}_1^{n-1}(\tau x, \eta)$. Similarly, the inequality $-(\sum_{j=0}^{s_1}(f(\tau^{j+1}x) - \alpha)) < -(f(\tau x) - \alpha)$ is imposible for $s_1 = 0$ and for the other possible values of s_1 the inequality contradicts $P \in \mathcal{P}_1^{n-1}(\tau x, \eta)$.
Define now $P'' = \{s_0 = -1 < t_0 = 0 \leq s_1' < t_1' \leq \ldots < t_m' \leq n\}$, notice that $|P''| = |P'| + 1 = w_{\eta,\alpha,n-1}(\tau x) + 1$; therefore, to finish the proof we should check that $P'' \in \mathcal{P}_0^n(x, \eta')$. Given that, as indicated earlier, $P' \in \mathcal{P}_0^n(x, \eta')$, and $f(x) - \alpha - \eta' \geq 0$ we only need to check if $-(\sum_{j=0}^{s_1'}(f(\tau^j x) - \alpha)) + f(x) - \alpha - \eta' \geq 0$. But this follows from (17) as indicated in the following display:

$$\eta' \leq -(\sum_{j=0}^{s_1}(f(\tau^{j+1}x) - \alpha)) = -(\sum_{j=0}^{s_1'}(f(\tau^j x) - \alpha)) + f(x) - \alpha.$$

□

3. Integral Inequalities for Generalized Upcrossings

The following upper bound can be found in [3].

Theorem 1. *Given real numbers α, η ($\eta > 0$), τ a measure preserving transformation and A an invariant subset. Then if $\chi_A\ (f - \alpha)_+ \in L^1$:*

$$\int_A \eta\ w_{\eta,\alpha}(x)\ d\mu(x) \leq \int_A\ (f(x) - \alpha)_+\ d\mu(x). \tag{18}$$

Proof. It follows by integrating (10) (after multiplication times χ_A) and noticing that under our hypothesis $\chi_A(x)\lambda_{\eta,\alpha,n}(x) \in L^1$ for all n. □

Lemma 6. *Assume τ is a measure preserving transformation, $f \in L^1$ and that real numbers α, η ($\eta > 0$) are given. Then for each x for which $(\alpha + \eta) > \lim_{n\to\infty} A_n(f)(x)$ the following limit exists as a real number,*

$$\lim_{n\to\infty} \lambda_{\eta,\alpha,n}(x). \tag{19}$$

Proof. Consider x such that $(\alpha + \eta) > \lim_{n\to\infty} A_n(f)(x)$, we will show first that there are positive integers $t_1 > t_0$ such that

$$\sum_{j=0}^{t}(f(\tau^j x) - \alpha - \eta) < 0 \text{ for all } t \geq t_0 \text{ and} \tag{20}$$

$$\sum_{j=0}^{t}(f(\tau^j x) - \alpha - \eta) - \sum_{j=0}^{s}(f(\tau^j x) - \alpha) < 0 \text{ for all } t > s \geq t_0 \text{ and} \tag{21}$$

$$\sum_{j=0}^{t}(f(\tau^j x) - \alpha - \eta) \leq \sum_{j=0}^{t_0}(f(\tau^j x) - \alpha - \eta) \text{ for all } t > t_1. \tag{22}$$

Let $g(x) := \lim_{n\to\infty} A_n(f)(x)$. Take $\epsilon = \min\left(\frac{\eta}{2}, \frac{\alpha+\eta-g(x)}{2}\right)$ and t_0 such that $|A_t f(x) - g(x)| < \epsilon$ for all $t \geq t_0$, hence:

$$A_t f(x) - \alpha - \eta < \epsilon + g(x) - \alpha - \eta \leq \frac{g(x) - \alpha - \eta}{2} < 0. \tag{23}$$

This proves (20). Let now $t > s \geq t_0$, for convenience set $\eta' = (g(x) - \alpha)$ and notice that $\epsilon \leq \frac{(\eta-\eta')}{2}$. We consider two cases; first $\eta' \geq 0$, then:

$$\sum_{j=0}^{t}(f(\tau^j x) - \alpha - \eta) - \sum_{j=0}^{s}(f(\tau^j x) - \alpha) < \tag{24}$$

$$(t-s)\,(g(x)-\alpha) - (t+1)\;\eta + (t+s+2)\;\epsilon < \frac{(\eta-\eta')}{2}(2+2t) + (t+1)(\eta'-\eta) = 0.$$

Now consider $\eta' < 0$ then:

$$\sum_{j=0}^{t}(f(\tau^j x) - \alpha - \eta) - \sum_{j=0}^{s}(f(\tau^j x) - \alpha) < \tag{25}$$

$$(t-s)\,(g(x)-\alpha) - (t+1)\;\eta + (t+s+2)\;\epsilon < 2(t+1)\frac{\eta}{2} - (t+1)\eta = 0.$$

Hence (21) is proven. We now prove (22), define $t_1 = 3\;t_0$ and take $t > t_1 + 1$; to simplify the notation let $y = A_{t_0} f(x) - \alpha - \eta$ and $z = A_t f(x) - \alpha - \eta$. Due to $t > t_0$ we get $|y - z| \leq 2\;\epsilon$, so

$$|y/z| \leq \frac{2\;\epsilon}{|z|} + 1. \tag{26}$$

Moreover, from $|A_t f(x) - g(x)| < \epsilon$ we obtain $|z| > -\;\epsilon + |g(x) - \alpha - \eta| \geq \epsilon$, where the last inequality follows from our choice of ϵ. Hence (26) gives $|y/z| \leq \frac{2\;\epsilon}{|g(x)-\alpha-\eta| -\;\epsilon} + 1 \leq 3$. From (20) we know that $y < 0$ and $z < 0$ hence $(t_0 + 1)\;y \geq (t+1)\;z$ which is (22). Equations (21) and (22) prove $\lambda_{\eta,\alpha,n-1}(x) = \lambda_{\eta,\alpha,n}(x)$ for all $n > t_1 + 1$ and hence (19) is proven. □

The following lower bound is our main result.

Theorem 2. *Assume $f \in L^1$ and α, η $(\eta > 0)$ are given real numbers. Let τ be a measure preserving transformation and A an invariant subset with $\mu(A) < \infty$, then if $(\alpha + \eta) > lim_{n\to\infty} A_n f(x)$ on A:*

$$\int_A (f(x) - \alpha - \eta)_+ \; d\mu(x) \leq \int_A \eta \; w_{\eta,\alpha}(x) \; d\mu(x). \tag{27}$$

Proof. We will use the notation $g_n(x) = (\lambda_{\eta,\alpha,n}(x) - \lambda_{\eta,\alpha,n-1}(x))\;\chi_A(x)$. One can check that $0 \leq g_n(x) \leq (f(\tau^n x) - \alpha - \eta)_+\;\chi_A(x) \leq T^n(f(x) - \alpha - \eta)_+\;\chi_A(x)$. We show next that $h_n(x) = T^n(f(x) - \alpha - \eta)_+\;\chi_A(x)$ (with $h(x) = h_0(x) = (f(x) - \alpha - \eta)_+\;\chi_A(x)$) is a uniformly integrable sequence hence, $g_n(x)$ is also uniformly integrable. Given that there exists a constant $a > 0$, independent of n, such that $||h_n||_1 \leq a\;||f||_1$, to check for uniform integrability it is enough to verify that for all $\epsilon > 0$ there exists a constant K_ϵ which satisfies $\int_X (h_n - K_\epsilon)_+\;d\mu(x) < \epsilon$ for all n. To verify this last statement take $\epsilon > 0$ and find K_ϵ such that $\int_X (h - h \wedge K_\epsilon)\;d\mu(x) < \epsilon$. Then

(28)

$$\int_X (h_n - K_\epsilon)_+\;d\mu(x) = \int_X (h_n - h_n \wedge K_\epsilon)\;d\mu(x) = \int_X (T^n h - T^n h \wedge K_\epsilon)\;d\mu(x) \leq$$

$$\int_X T^n(h - h\wedge K_\epsilon)\, d\mu(x) = \int_X (h - h\wedge K_\epsilon)\, d\mu(x) < \epsilon.$$

Multiply (7) by $\chi_A(x)$ and integrate to obtain

(29)

$$\int_A (f(x)-\alpha-\eta)_+\, d\mu(x) \leq \int_A w_{\eta,\alpha,n-1}(x)\, \eta\, d\mu(x) + \int_A (\lambda_{\eta,\alpha,n}(x) - \lambda_{\eta,\alpha,n-1}(x))\, d\mu(x).$$

We will use Theorem 5 applied to the uniformly integrable sequence $g_n(x)$. Notice that from Lemma 6 we have $\lim_{n\to\infty} g_n(x) = 0$ *a.e.* on A. Also $\lim_{n\to\infty} \int_A w_{\eta,\alpha,n-1}(x) = \int_A w_{\eta,\alpha}(x)$ because of Lebesgue's monotone convergence theorem. Hence taking $\lim_{n\to\infty}$ in (29) and using (51) gives Equation (27). ☐

Remark 1. *The condition $\alpha + \eta > lim_{n\to\infty} A_n f(x)$ seems to be needed because we are dealing with upcrossings, it can be removed once we introduce downcrossings as we do in Section 5. For the case when τ is ergodic and $\mu(X) < \infty$ the condition becomes $(\alpha+\eta)\mu(X) > \int_X f$.*

4. Generalized Upcrossings and Spatial Oscillations

In this section we clarify the geometric meaning of generalized upcrossings. We do this by establishing some relationships with the usual (geometric) upcrossings and with oscillations (or jumps).

Definition 4. Upcrossings:
Given a function $f(x)$, an integer $n \geq 0$, real numbers α, $\eta > 0$ and $x \in X$ define

$$U_{\eta,\alpha,n}(x) = \max\{k : \zeta = (u_r, v_r)_{r=1,\ldots,k.}\}, \tag{30}$$

where the sequence ζ satisfies,

$$-1 \leq u_1 < v_1 < u_2 < \ldots < v_k \leq n \tag{31}$$

$$A_{u_r} f(x) \leq \alpha \quad \text{and} \quad A_{v_r} f(x) \geq (\alpha + \eta), \tag{32}$$

for $r = 1, \ldots, k$. The sequence ζ will be called an *n- upcrossing sequence at x* and the space of these sequences denoted by $\mathcal{U}_0^n(x,\eta,\alpha)$. The function $U_{\eta,\alpha}(x) = \lim_{n\to\infty} U_{\eta,\alpha,n}(x)$ will be referred to as the number of upcrossings through the interval $[\alpha, \alpha+\eta]$ (see [3]).

The following simple proposition is a key motivation for the study of upper bounds for $w_{\eta,\alpha}$.

Proposition 1. *We have $\mathcal{U}_0^n(x,\eta,\alpha) \subseteq \mathcal{P}_0^n(x,\eta,\alpha)$, hence:*

$$U_{\eta,\alpha,n}(x) \leq w_{\eta,\alpha,n}(x). \tag{33}$$

Notice that in general $\lim_{\eta\to 0} U_{\eta,\alpha}(x) < \infty$ unless, for example, in the ergodic case, $\alpha = \int f$. The following Proposition gives information on what happens to $w_{\eta,\alpha}(x)$ as $\eta \to 0$.

Proposition 2. *Assume τ is an ergodic transformation. For any $p \geq 0$ define the measurable sets $A_{p,\infty} = \{x | \liminf_{\eta\to 0} \eta^p\; w_{\eta,\alpha}(x) = \infty\}$ then:*

$$\mu(A_{p,\infty}) = \mu(X) \text{ or } \mu(A_{p,\infty}) = 0, \tag{34}$$

if, in addition, $(f-\alpha)_+ \in L^1$ then for any $p \geq 1$

$$\mu(A_{p,\infty}) = 0. \tag{35}$$

Proof. Consider $p = 0$ first and notice that $\liminf_{\eta\to 0} w_{\eta,\alpha}(x) = \lim_{\eta\to 0} w_{\eta,\alpha}(x)$ in this case. From Lemma 4 and Lemma 5 it follows that $\tau^{-1}A_{0,\infty} = A_{0,\infty}$, therefore (34) follows from ergodicity of τ. Consider now $p > 0$, for each integer M define the sets $A_{p,M} = \{x | \liminf_{\eta\to 0} \eta^p\ w_{\eta,\alpha}(x) > M\}$. From Lemma 4 and Lemma 5 it follows that

$$A_{p,M} \subseteq \tau^{-1}(A_{p,M}) \subseteq A_{p,M/2^p}. \tag{36}$$

Notice that for any k $A_{p,\infty} = \cap_{M=k}^{\infty} A_{p,M}$, hence (36) gives $A_{p,\infty} \subseteq \tau^{-1}A_{p,\infty} \subseteq A_{p,\infty}$. This proves $A_{p,\infty}$ is an invariant subset, therefore (34) follows from ergodicity of τ. To prove (35) we just need to consider $p = 1$, assume $\mu(A_{1,\infty}) = \mu(X)$, applying Fatou's theorem to (18) and to the invariant set $A = A_{1,\infty}$ in that equation, we obtain $\liminf_{\eta\to\infty}\ \eta w_{\eta,\alpha}(x) < \infty$ a.e., hence $\mu(A_{1,\infty}) = 0$. □

Remark 2. *From the above proposition and Theorem 2 it is natural to expect that under rather general conditions we may actually have $\mu(A_{p,\infty}) = \mu(X)$ for $0 \le p < 1$.*

Definition 5. Jumps:
Given a function $f(x)$, a fixed integer $n \ge 0$, a real number $\eta > 0$ and $x \in X$, define

$$J_{\eta,n}(x) = \max\{k : \xi = (t_r)_{r=0,\dots,k}\},$$

where ξ satisfies:

$$-1 \le t_0 < t_1 < t_2 < \dots < t_k \le n$$

and

$$|A_{t_{r+1}} f(x) - A_{t_r} f(x)| \ge \eta, \quad \textit{for all } r = 0,\dots,k-1. \tag{37}$$

Also define

$$J_\eta(x) = \sup\{J_{\eta,n}(x) : n \ge 0\}$$

the function J_η will be referred to as the number of η-jumps.

Taken together, the proof of item (3) in the next lemma and Proposition 1 give a rather complete picture of the geometric meaning of generalized upcrossings.

Lemma 7. *For given real numbers α, η $(\eta > 0)$ and integer $n \ge 0$ we have:*

1. *If $P = (s_1, t_1 \dots, s_m, t_m) \in \mathcal{P}_0^n(x)$ then $t_i < s_{i+1}$ for $i = 1,\dots,m-1$.*
2. *$w_{\eta,\alpha,n+1}(x) = w_{\eta,\alpha,n}(x)$ or $w_{\eta,\alpha,n+1}(x) = w_{\eta,\alpha,n}(x) + 1$.*
3. *If $P = (s_1, t_1 \dots, s_m, t_m) \in \mathcal{P}_0^n(x)$ with $m = w_{\eta,\alpha,n}(x)$ then there exists a sequence $-1 \le \theta_0 < \theta_1 < \theta_2 < \dots < \theta_m \le t_m$ such that*

$$|A_{\theta_{i+1}} f(x) - A_{\theta_i} f(x)| \ge \frac{\eta}{2}, \quad \textit{for all } i = 0,\dots,m-1. \tag{38}$$

Proof.
(1) If there exists i such that $t_i = s_{i+1}$ we have that $b_{t_i} - a_{s_i} = -(t_i + 1)\eta < 0$.
(2) We suppose that $w_{\eta,\alpha,n+1}(x) > w_{\eta,\alpha,n}(x)$. Let $P = (s_1, t_1 \dots, s_r, t_r) \in \mathcal{P}_2^{n+1}(x)$. From Lemma 1 $(s_1, t_1 \dots, s_{r-1}, t_{r-1}) \in \mathcal{P}_0^n(x)$ and $w_{\eta,\alpha,n}(x) \ge r - 1$.
(3) We will prove it by induction on n, by the previous item we may assume $w_{\eta,\alpha,n}(x) = w_{\eta,\alpha,n-1}(x) + 1 = m$. Let $P = (s_1, t_1, \dots, s_m, t_m) \in \mathcal{P}_0^n(x)$, this implies

$$b_{t_i}(x) - a_{s_i}(x) \ge 0 \text{ for } i = 1,\dots,m \tag{39}$$

and

$$b_{t_i}(x) - a_{s_{i+1}}(x) \geq 0 \text{ for } i = 1, \ldots, m-1. \tag{40}$$

We will first prove that $A_{t_{m-1}} f(x) - A_{s_m} f(x) \geq \eta$ or $A_{t_m} f(x) - A_{s_m} f(x) \geq \eta$. To this end we first assume $A_{s_m} f(x) > \alpha$ and $A_{t_{m-1}} f(x) - A_{s_m} f(x) < \eta$, we then have :

$$0 > t_{m-1}(A_{t_{m-1}} f(x) - \alpha - \eta) - t_{m-1}(A_{s_m} f(x) - \alpha) \geq \tag{41}$$
$$t_{m-1}(A_{t_{m-1}} f(x) - \alpha - \eta) - s_m(A_{s_m} f(x) - \alpha) = b_{t_{m-1}}(x) - a_{s_m}(x)$$

which contradicts (40).
Now, we assume $A_{s_m} f(x) \leq \alpha$ and $A_{t_m} f(x) - A_{s_m} f(x) < \eta$, so :

$$0 > t_m(A_{t_m} f(x) - \alpha - \eta) - t_m(A_{s_m} f(x) - \alpha) \geq \tag{42}$$
$$t_m(A_{t_m} f(x) - \alpha - \eta) - s_m(A_{s_m} f(x) - \alpha) = b_{t_m}(x) - a_{s_m}(x)$$

which contradicts (39).
Since $(s_1, t_1, \ldots, s_{m-1}, t_{m-1}) \in \mathcal{P}_0^{n-1}(x)$, by inductive hypothesis, there exists a sequence $-1 \leq \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{m-1} \leq t_{m-1}$ such that (38) holds. We need now to define θ_m, it is enough to study the following two cases. i) $A_{t_{m-1}} f(x) - A_{s_m} f(x) \geq \eta$: then if $\theta_{m-1} = t_{m-1}$ we take $\theta_m = s_m < t_m$. Consider then $\theta_{m-1} < t_{m-1}$. Now if $|A_{t_{m-1}} f(x) - A_{\theta_{m-1}} f(x)| < \frac{\eta}{2}$ it follows that $|A_{\theta_{m-1}} f(x) - A_{s_m} f(x)| \geq \frac{\eta}{2}$ so we can set again $\theta_m = s_m$. Otherwise, if $|A_{t_{m-1}} f(x) - A_{\theta_{m-1}} f(x)| \geq \frac{\eta}{2}$ we take $\theta_m = t_{m-1}$. ii) $A_{t_m} f(x) - A_{s_m} f(x) \geq \eta$: consider then $|A_{t_m} f(x) - A_{\theta_{m-1}} f(x)| < \frac{\eta}{2}$ in which case $|A_{s_m} f(x) - A_{\theta_{m-1}} f(x)| \geq \frac{\eta}{2}$, clearly $\theta_{m-1} < s_m$ hence we can take $\theta_m = s_m < t_m$. If $|A_{t_m} f(x) - A_{\theta_{m-1}} f(x)| \geq \frac{\eta}{2}$ we take $\theta_m = t_m$.

□

The following corollary, which follows from item (3) in Lemma 7, permits to transfer all of our lower bound inequalities for $w_{\eta,\alpha,n}(x)$ to lower bound inequalities for $J_{\eta,n}(x)$. From results in [4] it is expected that the so obtained inequalities will not be tight in general.

Corollary 2.

$$\sup_{\alpha} (w_{\eta,\alpha,0}(x)) \leq J_{\eta,0}(x),$$
$$\sup_{\alpha} (w_{\eta,\alpha,n}(x)) \leq J_{\eta',n}(x) \textit{ for all } \eta' \leq \eta/2 \textit{ and } n \geq 1.$$

5. Integral Inequalities for Generalized Downcrossings

Given our techniques, it turns out to be relevant to consider generalized downcrossings. They can easily be related to previous introduced quantities and to geometric downcrossings but we will not need to spell out those relationships here.
For real numbers α, η ($\eta > 0$) and given x we especialize the given sequences $\{a_i\}$ and $\{b_i\}$ in Definition 2 as follows,

$$b_t = b_t^d(x) = b_{t,\alpha}^d(x) = -\sum_{j=0}^{t} (f_j(x) - \alpha)$$

and

$$a_s = a_s^d(x) = a_{s,\eta,\alpha}^d(x) = -\sum_{j=0}^{s} (f_j(x) - \alpha - \eta).$$

The set $\mathcal{P}_0^n$ in Definition 2 especializes to the set of n-generalized downcrossing sequences denoted by $\mathcal{P}_{d,0}^n(x,\eta,\alpha)$.
For a nonvoid admissible sequence $P = (s_1, t_1, \dots, s_m, t_m)$, we define:

$$S_d(P)(x) = \sum_{i=1}^{|P|} \left(b_{t_i}^d(x) - a_{s_i}^d(x) \right). \tag{44}$$

Let $\mathcal{P}_{d,1}^n(x)$ be the set of n-admissible sequences P with $S_d(P)(x)$ maximal in $\mathcal{P}^n$. Let $\mathcal{P}_{d,2}^n(x)$ be the set of sequences P in $\mathcal{P}_{d,1}^n(x)$ with $|P|$ maximal. Similarly as we did for upcrossings we introduce the following notation for generalized downcrossings.

Definition 6. For given integer $n \geq 0$, we define

$$\lambda_{\eta,\alpha,n}^d(x) = \max_{P \in \mathcal{P}^n} S_d(P)(x) = S_d(P_1)(x), \tag{45}$$

where P_1 is any element in $\mathcal{P}_{d,1}^n(x)$. The (maximum) *number of n-generalized downcrossings* at x is given by:

$$w_{\eta,\alpha,n}^d(x) = \max\{|P| : \text{where } P \text{ is an n-generalized downcrossing sequence at } x\}. \tag{46}$$

Also, define the number of generalized downcrossings at x by $w_{\eta,\alpha}^d(x) = \lim_{n\to\infty} w_{\eta,\alpha,n}^d(x)$.

With the above definitions Lemma 1 is inmediately applicable. Analogous results to Corollary 1, Lemma 2 and Lemma 3 can be obtained for the quantities defined above. Finally, we have the following dual theorems for generalized downcrossings.

Theorem 3. *Assume $\chi_A\ (\alpha + \eta - f)_+ \in L^1$, where A is an invariant subset with respect to τ, a measure preserving transformation. If α, η ($\eta > 0$) are given real numbers then:*

$$\int_A \eta\ w_{\eta,\alpha}^d(x)\ d\mu(x) \leq \int_A\ (\alpha + \eta - f(x))_+\ d\mu(x). \tag{47}$$

Theorem 4. *Assume $f \in L^1$ and α, η ($\eta > 0$) are given real numbers. Let τ be a measure preserving transformation and A an invariant subset with $\mu(A) < \infty$, then if $(\alpha + \eta) \leq lim_{n\to\infty} A_n f(x)$ on A:*

$$\int_A\ (\alpha - f(x))_+\ d\mu(x) \leq \int_A \eta\ w_{\eta,\alpha}^d(x)\ d\mu(x). \tag{48}$$

By combining the results for generalized downcrossings and upcrossings we can remove the hypothesis on α. To do this define $m_{\eta,\alpha}(x) = \max\left(w_{\eta,\alpha}^d(x),\ w_{\eta,\alpha}(x)\right)$. To simplify the hypothesis of the next corollary we will take X to have finite measure.

Corollary 3. *Let $f \in L^1$, $\mu(X) < \infty$, α, η real numbers ($\eta > 0$). If τ is a measure preserving transformation and A an invariant set, then:*

$$\min\left(\int_A (f - \alpha - \eta)_+, \int_A (\alpha - f)_+ \right) \leq \int_A\ \eta\ m_{\eta,\alpha}. \tag{49}$$

Appendix A. Background Material

Here we mention a known result used in the paper, it is an extension of Lebesgue's dominated convergence theorem in the setting of uniformly integrable functions.

Definition 7. A sequence of measurable functions g_n in a finite measure space (A, μ) is said to be uniformly integrable if:

$$\int_{\{|g_n| \geq c\}} |g_n| \, d\mu \to 0 \text{ as } c \to \infty, \text{ uniformly in } n. \tag{50}$$

Theorem 5. *In the above setting we have (see* [1] *pg. 295): If $g_n(x) \to g(x)$ a.e. then g is integrable and:*

$$\int_A \lim_{n\to\infty} g_n \, d\mu = \lim_{n\to\infty} \int_A g_n \, d\mu \tag{51}$$

Acknowledgement: We want to thank Dr. John Baxter for the present version of Lemma 1 and his continuous encouragement.

References

[1] R.P. Ash, Real Analysis and Probability, *Academic Press*, 1972.

[2] Bishop, Errett,*A constructive ergodic theorem*, Journal of Mathematics and Mechanics **17** (1968), 631-639.

[3] Bishop, Errett, *Foundations of Constructive Analysis*, McGraw-Hill, New York 1967.

[4] P.J. Catuogno and S.E. Ferrando, *Pointwise asymptotics for the jumps of ergodic averages.* New York Journal of Mathematics, **Vol. 7** (2001), 59-69.

[5] R.L. Jones, J. Rosenblatt, *Reverse inequalities*, The Journal of Fourier Analysis and Applications, **Vol. 6**, Issue 3 (2000), 325-341.

Department of Mathematics, Physics and Computer Science, Ryerson Polytechnic University, Toronto, Ontario M5B 2K3, Canada.

E-mail address: ferrando@acs.ryerson.ca

Departamento de Matematicas. Facultad de Ciencias Exactas y Naturales. Universidad Nacional de Mar del Plata. Funes 3350, Mar del Plata 7600, Argentina.

E-mail address: pedrojc@mdp.edu.ar

E-mail address: algonzal@mdp.edu.ar